$$\sum_{i=1}^{k} \frac{1}{\sqrt[2n]{b_i}} \geq k \sqrt[k]{\prod_{j=1}^{k} \frac{1}{\sqrt[2n]{b_i}}} = k \sqrt[2nk]{\frac{1}{\prod_{j=1}^{k} b_i}} = k \sqrt[2nk]{\frac{1}{A^{k-1}}} \geq k$$

because $A \leq 1$. Thus,

$$\sum_{cyc} \sqrt[n]{a_1 + \frac{1}{\prod_{j=1}^{k} a_j}} \geq k \sqrt[n]{2}.$$

Arkady Alt

W28. (**Solution by the proposer.**) The roots of characteristic polynomial equation $x^2 - x + \frac{1}{2} = 0$ are $\alpha = \frac{1+i}{2}$ and $\beta = \frac{1-i}{2}$, and taking into account the initial values of the sequence, it follows that $x_n = \alpha^n + \beta^n$, so the proposed series reads may be written as

$$\sum_{n=1}^{\infty} \frac{\alpha^n + \beta^n}{n+2}.$$

The generating function for $(x_n)_{n\geq 0}$ is $F(x) = \sum_{n=0}^{\infty} x_n x^n = \frac{2-x}{\frac{x^2}{2} - x + 1}$.

Funtion $F(z)$ is analytic for $z \in C$ with $|z| < \sqrt{2}$.

To find the proposed series $\sum_{n=1}^{\infty} \frac{x_n}{n+2} = \sum_{n=1}^{\infty} \frac{\alpha^n + \beta^n}{n+2}$, we consider the function $x \cdot F(x) = \frac{2x - x^2}{\frac{x^2}{2} - x + 1}$, and its integral

$$\int \frac{2x - x^2}{\frac{x^2}{2} - x + 1} dx = -2\left(x + 2\tan^{-1}(1-x)\right) = G(x)$$

so $\sum_{n=1}^{\infty} \frac{x_n x^{n+2}}{n+2} = G(x) - G(0) = \pi - 2\left(x + 2\tan^{-1}(1-x)\right)$.

Therefore, $\sum_{n=1}^{\infty} \frac{x_n}{n+2} = G(1) - G(0) = \pi - 2\left(1 + 2\tan^{-1}(1)\right) = \pi - 2$.

Second solution. Note that

$$x_{n+2} = x_{n+1} - \frac{1}{2} x_n \iff$$

$$\iff \left(\sqrt{2}\right)^{n+2} x_{n+2} - 2\cdot\frac{1}{\sqrt{2}}\left(\sqrt{2}\right)^{n+1} x_{n+1} + \left(\sqrt{2}\right)^{n} x_n = 0 \iff$$

$$\iff \left(\sqrt{2}\right)^{n+2} x_{n+2} - 2\cdot\cos\frac{\pi}{4}\left(\sqrt{2}\right)^{n+1} x_{n+1} + \left(\sqrt{2}\right)^{n} x_n = 0 \iff$$

$$\iff \left(\sqrt{2}\right)^{n} x_n = c_1\cos\frac{n\pi}{4} + c_2\sin\frac{n\pi}{4}, n\in\mathbb{N}\cup\{0\}.$$

Since $x_0 = 2 \implies c_1 = 2$ and
$x_1 = 1 \iff (\sqrt{2}) = 2\cos\frac{\pi}{4} + c_2\sin\frac{\pi}{4} \iff c_2 = 0$ then
$x_n = \frac{2\cos\frac{n\pi}{4}}{\left(\sqrt{2}\right)^{n}}, n\in\mathbb{N}\cup\{0\}$ and

$$\sum_{n=1}^{\infty}\frac{x_n}{n+2} = \sum_{n=0}^{\infty}\frac{x_n}{n+2} - 1 = 2\sum_{n=0}^{\infty}\frac{\cos\frac{n\pi}{4}}{\left(\sqrt{2}\right)^{n}(n+2)} - 1 =$$

$$= 4\sum_{n=0}^{\infty}\frac{\cos\frac{n\pi}{4}}{\left(\sqrt{2}\right)^{n+2}(n+2)} - 1$$

Let $S(x) := \sum\limits_{n=0}^{\infty}\frac{\cos\frac{n\pi}{4}x^{n+2}}{n+2}$, where $x\in(0,1)$. Then
$\sum\limits_{n=1}^{\infty}\frac{x_n}{n+2} = 4S\left(\frac{1}{\sqrt{2}}\right) - 1$
and $S'(x) = x\sum\limits_{n=0}^{\infty}x^n\cos\frac{n\pi}{4}$, where series $\sum\limits_{n=1}^{\infty}\cos\frac{n\pi}{4}x^n$ converges absolutely

$$\sum_{n=1}^{\infty}\left|\cos\frac{n\pi}{4}x^n\right| \le \sum_{n=1}^{\infty}|x|^n = \frac{|x|}{1-|x|}$$

Noting that

$$\sum_{n=0}^{\infty}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)x^n = \sum_{n=0}^{\infty}e^{\frac{in\pi}{4}}x^n = \sum_{n=0}^{\infty}\left(e^{\frac{i\pi}{4}}x\right)^n =$$

$$= \frac{1}{1-e^{\frac{i\pi}{4}}x}$$

(since $\left|e^{\frac{i\pi}{4}}x\right| < 1$) and

$$\frac{1}{1-e^{\frac{i\pi}{4}}x}=\frac{1}{1-\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)x}=\frac{1}{1-\frac{x}{\sqrt{2}}-i\frac{x}{\sqrt{2}}}=$$

$$=\frac{\sqrt{2}}{\sqrt{2}-x-ix}=\frac{\sqrt{2}\left(\sqrt{2}-x+ix\right)}{\left(\sqrt{2}-x\right)^2+x^2}$$

we obtain

$$S'(x)=x\sum_{n=0}^{\infty}\cos\frac{n\pi}{4}x^n=\frac{\sqrt{2}\left(\sqrt{2}-x\right)x}{\left(\sqrt{2}-x\right)^2+x^2}=-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}\left(x^2-\sqrt{2}x+1\right)}.$$

Since $\sum\limits_{n=1}^{\infty}\frac{x_n}{n+2}=4S\left(\frac{1}{\sqrt{2}}\right)-1$ and

$$S\left(\frac{1}{\sqrt{2}}\right)=\int_0^{1/\sqrt{2}}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}\left(x^2-\sqrt{2}x+1\right)}\right)dx=\frac{\pi}{4}-\frac{1}{2}$$

then

$$\sum_{n=1}^{\infty}\frac{x_n}{n+2}=4\left(\frac{\pi}{4}-\frac{1}{2}\right)-1=\pi-3.$$

Arkady Alt

W29. (Solution by the proposer.) Since $x+y+z=1$, the given inequality may be written as

$$\sum_{cyclic}\sqrt{\frac{x^3+1}{x^2-x+1}}\le 3\sqrt{2}.$$

Now, using that $x^3+1=(x+1)(x^2-x+1)$, the inequality becomes $\sum\limits_{cyclic}\sqrt{x+1}\le 3\sqrt{2}$. Finally, since function $f(x)=\sqrt{x+1}$ is concave because $f''(x)=\frac{-1}{4(x+1)^{3/2}}<0$, by Jensen's inequality, we have

$$\frac{\sum\limits_{cyclic}\sqrt{x+1}}{3}\le\sqrt{\frac{x+y+z}{3}+1}=\sqrt{2}$$